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Limiting geometries of two circular Maldacena-Wilson loop operators

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Abstract: We further analyze a recent perturbative two-loop calculation of the expectation value of two axi-symmetric circular Maldacena-Wilson loops in $\mathcal{N} = 4$ gauge theory. Firstly, it is demonstrated how to adapt the previous calculation of anti-symmetrically oriented circles to the symmetric case. By shrinking one of the circles to zero size we then explicitly work out the first few terms of the local operator expansion of the loop. Our calculations explicitly demonstrate that circular Maldacena-Wilson loops are non-BPS observables precisely due to the appearance of unprotected local operators. The latter receive anomalous scaling dimensions from non-ladder diagrams. Finally, we present new insights into a recent conjecture claiming that coincident circular Maldacena-Wilson loops are described by a gaussian matrix model. We report on a novel, supporting two-loop test, but also explain and illustrate why the existing arguments in favor of the conjecture are flawed.

Keywords: Duality in Gauge Field Theories, AdS-CFT Correspondance, Extended Supersymmetry, Matrix Models.

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1. Introduction and conclusions

Wilson loops are interesting non-local observables in gauge theories. They are functionals of macroscopic space-time contours and are widely suspected to be the proper observables to describe the strong coupling physics of Yang-Mills theories. While they have been quite useful as order parameters in lattice gauge theory, Wilson loops have unfortunately been rather difficult to work with in continuum gauge theories. One of the chief reasons is that they are plagued by various infinities whose removal obscures the seeming simplicity of their bare equations of motion. The form of the renormalized, non-perturbative loop equations remains unknown.

For smooth loop contours the mentioned infinities can be divided into two classes: in addition to the usual quantum field theoretic "bulk" divergences stemming from Feynman diagrams containing internal loops there are further "boundary" divergences due to contact interactions on the boundaries of the Wilson loops.

Recently, a much more transparent situation has begun to emerge in maximally supersymmetric gauge theory. In [1] a modification of the usual Wilson loop operators has been proposed. Here the loops are not only coupled to the gauge field but in addition to the six scalar fields of the model. This modification significantly improves the just mentioned divergence problems: The boundary divergences of individual Feynman graphs are either absent or, even more interestingly, cancel against bulk divergences. This phenomenon has so far been observed in a number of one-loop and two-loop calculations [2, 3]. However, a general proof of the perturbative, let alone non-perturbative, finiteness of these novel Maldacena-Wilson loop operators is still lacking to date. Finding such a proof would yield an infinite set of finite geometric probes of a four-dimensional quantum field theory.

A further — actually the initial — motivation for being interested in these operators is that they are conjectured to be directly related, at strong coupling and in the so-called planar limit, to certain classical supergravity solutions in a special background [1]. Finally, there is even hope that for some special contours such as a circle these loops might be exactly calculable [2, 4], which, if proven, might lead to non-trivial analytic tests of the proposed supergravity-gauge correspondence for these macroscopic observables. To date the correspondence [5, 6, 7] has only been rigorously tested on the level of certain local operators.

In [3] the first two-loop perturbative study of Maldacena-Wilson loops was reported. The specific situation analyzed consisted in two axi-symmetric circular contours of arbitrary radii and distance. The motivation of [3] was to perform a two-loop test of the finiteness properties, as well as to establish that the static potential, obtained by sending the radii to infinity, receives contributions from non-ladder diagrams, in contradistinction to an accidental one-loop cancellation of interactive diagrams. For related work on Maldacena-Wilson loops see [8]–[13].

In the present paper the results of [3] are applied to two further situations of physical interest. This requires a rather straightforward extension of our previous results, which were derived for circles of opposite orientation, to also include the equal-orientation case (see section 2). We find that it simply corresponds to formally flipping the relative sign of the two radii. In section 3 we then study, for both orientations, the limit where either one of the two circles shrinks to zero size. In this limit one expects that the shrinking loop should be expandable in local operators of increasing scaling dimension. This allows to extract term-by-term the expectation value of local operators with the remaining "large" circle. It is shown that our previous results contain enough information to extract the one-loop anomalous dimension of the fields of approximate scaling dimension two and three. In particular we recover the known one-loop anomalous dimension of the Konishi field, which is the lowest (~ 2) dimension unprotected operator of the theory. More importantly, it is reassuring that the anomalous scaling turns out to be precisely due to the *non-ladder diagrams*. In turn, it is seen that the unprotected operators appearing in the operator expansion are responsible for the global non-BPS nature of closed Maldacena-Wilson contours. We also comment on the appearance of unprotected operators of classical dimension three and four and their correspondence to the supergravity limit.

The second application, presented in section 4, involves the limit of equal but finite radii and vanishing distance of the equally oriented circles.[1] This allows us to perform the first two-loop test of a conjecture due to Drukker and Gross [4] (see also [2]), which holds that multiple coincident circular Maldacena-Wilson loops can be described by a gaussian matrix model. We find that all connected non-ladder diagrams cancel to $\mathcal{O}(g^6)$ for the case of two circles, in line with the conjecture.

We nevertheless feel the need to add some cautionary observations. The fact that so far the gaussian matrix model description appears to be valid might be a purely accidental low-order phenomenon. At any rate we argue that no real arguments exist that vertex diagrams should not contribute to the circle, and that the analysis presented in [4], which involves a conformal map of an infinite line to a circle, does not directly apply to vertex graphs. As an illustration we consider a particular two-loop non-ladder graph for a single circle and demonstrate that it is, just like the ladder graphs, zero for the line but perfectly finite and non-zero for the circle. It might still be true that the *sum* of all non-ladder

[1] This is the identical to the "static potential" limit considered in [3] except that there the orientation was anti-symmetric.

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diagrams cancels, but we so far lack any argument for this. Short of a general proof, it would clearly be desirable to complete the full two-loop calculation for a single circle.

2. Two loops of equal orientation

In [3] a two loop perturbative calculation of the (connected) expectation value of two circular Maldacena-Wilson loops of *opposite* orientation was performed. Here we shall be interested in the scenario with two loops of *equal* orientation. That is we consider the connected correlator of two Maldacena-Wilson loops

$$\langle \mathcal{W}(C_1)\,\mathcal{W}(C_2)\rangle_c = \langle \mathcal{W}(C_1)\,\mathcal{W}(C_2)\rangle - \langle \mathcal{W}(C_1)\rangle\,\langle \mathcal{W}(C_2)\rangle\,. \tag{2.1}$$

The Maldacena-Wilson loop operator is defined by

$$\mathcal{W}[C] = \mathrm{Tr}\,\mathrm{P}\exp\left[\oint_C d\tau\,\left(iA_\mu(x)\dot{x}^\mu + \Phi_I(x)\theta^I|\dot{x}|\right)\right]. \tag{2.2}$$

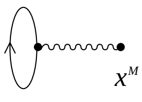
Here θ^I is a point on the unit five-sphere, i.e. $\theta^I\theta^I = 1$, and $x^\mu(\tau)$ parameterizes the curve C. We take the curves C_1 and C_2 to be two parallel, axi-symmetric circles of equal orientation and, respectively, radii R_1 and R_2 separated by a distance h

$$\begin{aligned} x^\mu(\tau) &= (R_1\,\cos\tau, R_1\,\sin\tau,\,0\,,\,0) \\ y^\mu(\sigma) &= (R_2\,\cos\sigma, R_2\,\sin\sigma,\,h\,,\,0) \qquad \tau,\sigma\in[0,2\pi]\,. \end{aligned} \tag{2.3}$$

All our conventions follow [3]. For calculational purposes it is useful to go ten dimensional notation $M = (\mu, I)$, $\mathcal{A}^a_M[x] = (A^a_\mu(x), \Phi^a_I(x))$ and $\dot{x}(\tau) = (\dot{x}^\mu(\tau), \theta^I\,|\dot{x}^\mu|)$ with the combined gluon-scalar propagator in 2ω dimensions

$$\langle \mathcal{A}^a_M[x]\,\mathcal{A}^b_N[y]\rangle = g^2\,\delta^{ab}\,\delta_{MN}\,\frac{\Gamma(\omega-1)}{4\pi^\omega}\,\frac{1}{[(x-y)^2]^{\omega-1}} \tag{2.4}$$

in Feynman gauge where x and y are points in 2ω dimensional Euclidean space. As a matter of fact the results of [3] may be directly translated to our new scenario of circles of equal orientation.[2] To do so consider the integrated Maldacena-Wilson loop associated with R_2 with an open leg at the point x^μ



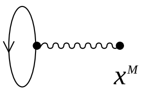
$$\begin{aligned} &= \int_0^{2\pi} d\sigma\,\dot{y}(\sigma)\,\langle\,\mathcal{A}^a_M[y(\sigma)]\,\mathcal{A}^b_N[x]\,\rangle \\ &= \frac{g^2\,\delta^{ab}\,\Gamma(\omega-1)}{4\pi^\omega}\int_0^{2\pi} d\sigma\,\frac{(\dot{y}^\mu(\sigma), \theta^I R_2)}{[(y(\sigma)-x)^2]^{\omega-1}} \\ &= \frac{g^2\,\delta^{ab}\,\Gamma(\omega-1)}{4\pi^\omega}\int_0^{2\pi} d\sigma\,\frac{R_2\,(-\cos\sigma\,\sin\phi, \cos\sigma\,\cos\phi,\,0\,,\,0\,,\theta^I)}{(A^2-2R_2\,\rho\,\cos\sigma)^{\omega-1}}\,, \end{aligned} \tag{2.5}$$

[2]We wish to thank N. Drukker for independently suggesting that our calculation could be easily modified to also cover this situation. He also, correctly, proposed that the equal orientation case could be obtained by simply flipping the sign of one of the two radii.

where we have introduced polar coordinates $x_1 = \rho\cos\phi$, $x_2 = \rho\sin\phi$ and $A^2 = R_2^2 + \rho^2 + (x_3 - h)^2 + x_4^2$. Moreover a shift in $\sigma \to \sigma + \phi$ was performed. Now consider the same graph with a circle of opposite orientation, parametrized by

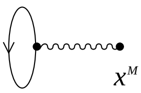
$$\bar{y}^\mu(\sigma) = (R_2\,\cos\sigma, -R_2\,\sin\sigma,\, h\,,\, 0)\,. \tag{2.6}$$

Performing the analogous manipulations as in (2.5) one arrives at

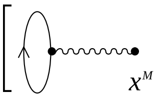


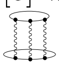
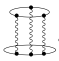
$$\begin{aligned} &= \frac{g^2\,\delta^{ab}\,\Gamma(\omega-1)}{4\pi^\omega}\int_0^{2\pi} d\sigma\,\frac{(\dot{\bar{y}}^\mu(\sigma), \theta^I R_2)}{[(\bar{y}(\sigma)-x)^2]^{\omega-1}} \\ &= \frac{g^2\,\delta^{ab}\,\Gamma(\omega-1)}{4\pi^\omega}\int_0^{2\pi} d\sigma\,\frac{R_2\,(\cos\sigma\,\sin\phi, -\cos\sigma\,\cos\phi,\, 0\,,\, 0\,, \theta^I)}{(A^2 - 2R_2\,\rho\,\cos\sigma)^{\omega-1}}\,. \end{aligned} \tag{2.7}$$

Now clearly (2.7) may be transformed into (2.5) by first shifting the integration variable $\sigma \to \sigma + \pi$, thereafter sending $R_2 \to -R_2$ and multiplying the total graph by -1. We have hence shown that



$$= -\Big[\quad\Big]_{R_2 \to -R_2}\,. \tag{2.8}$$



This reasoning is applicable to all graphs where there is at least one circle with no occurrence of path ordering along the loop. By looking at the relevant graphs up to order g^6 in [3] we see that the rule (2.8) is applicable to all graphs except for the 3-ladder diagrams  and . A careful reanalysis of these two diagrams, however, reveals that also for these two path ordered diagrams the rule (2.8) continues to hold.

In fact one wonders whether the rule (2.8) holds in general, i.e. for any graph. However, we have not yet found a proof due to the path ordering problems.

In summary we now have the complete two loop ($\mathcal{O}(g^6)$) computation for two connected Maldacena-Wilson loops of equal orientation at our disposal. For this geometry we again observe the complete finiteness of the Maldacena-Wilson loop, as the subtle bulk-boundary cancellation of divergences of the self-energy and the IY graphs of [3] continues to hold due to the simple rule (2.8).

3. Local operator expansion

We now turn to the study of our geometry when one of the two circles shrinks to zero size, i.e. the limit $R_1 \to 0$ while keeping R_2 and h finite. In [3] we derived explicit analytic expressions for all ladder graphs but had to content ourselves with somewhat complicated if finite integral representations for the non-ladder diagrams. Remarkably, in the present limit our expressions simplify sufficiently and we are able to deduce explicit analytic results for all graphs. Quite generally, one expects that the shrinking loop may be represented by an infinite sum of local composite operators of increasing scaling dimension. The two loop results of [3] allow to extract informations on the leading lower dimensional operators appearing in this expansion and make contact with recent studies of local composite operators in $\mathcal{N} = 4$ gauge theory performed in the context of the AdS/CFT correspondence (see e.g. [14]–[19]).

Consider the following local operator expansion of a Maldacena-Wilson loop

$$\frac{W(C)}{\langle W\rangle} = \mathbb{1} + \sum_k c_k R^{\Delta_k} O^{(k)}(x)\,, \tag{3.1}$$

where $O^{(k)}$ denotes a local composite operator of scaling dimension Δ_k. Assume the special case where the loop is a circle with radius R and lies in the (x_1, x_2) plane centered at x. Then the loop is sufficiently symmetric to only partially break Lorentz invariance. Thus the operator content in (3.1) is classified by irreducible representations of $\mathrm{SO}(2)\times\mathrm{SO}(2)$, the unbroken subgroup of $\mathrm{SO}(4)$. Then every four-dimensional index μ naturally splits into i and $\bar{i}$, where $i = 1, 2$ and $\bar{i} = 3, 4$. Moreover the loop possesses a definite orientation captured by the two-form ϵ^{ij} with $\epsilon^{12} = 1$.

In the case where R is small compared to any other distance in the problem at hand the Maldacena-Wilson loop operator may be replaced by its local operator expansion. In particular, for the connected Green function of two parallel Wilson loops C_1 and C_2 with radii R_1 and R_2, separated along the 3-axis by a distance h, eq. (3.1) implies the following decomposition

$$\frac{\langle W(C_1)W(C_2)\rangle_c}{\langle W\rangle^2} = \sum_k c_k R_1^{\Delta_k} \frac{\langle O^{(k)}(0)W(C_2)\rangle}{\langle W\rangle^2} \tag{3.2}$$

assuming that $R_1 \ll R_2, h$. In perturbation theory the scaling dimension of an operator can be represented as

$$\Delta = \Delta^{(0)} + \Delta^{(1)} + \Delta^{(2)} + \cdots\,, \tag{3.3}$$

where $\Delta^{(0)}$ is the free field dimension and $\Delta^{(1)}$, $\Delta^{(2)}$ are anomalous dimensions at order g^2, g^4 and so on. Therefore every term R^{Δ} in eq. (3.2) produces logarithmic terms as

$$R^{\Delta} = R^{\Delta^{(0)}}\left(1 + \Delta^{(1)}\log R + \Delta^{(2)}\log R + \frac{1}{2}{\Delta^{(1)}}^2\log^2 R + \cdots\right) \tag{3.4}$$

quite similar to the perturbative logarithms appearing in correlation functions of local correlators.[3] Hence, singling out in the limit $R \to 0$ the coefficients of the divergent logarithms $\log R$ in the connected Green function of two Wilson loops will allow us to determine the one-loop anomalous dimensions of the operators appearing in (3.1).

Let us briefly comment on the relation of the field-theoretic behavior of the expansion (3.1) with supergravity predictions. All local gauge-invariant operators in the $\mathcal{N} = 4$ super Yang-Mills theory are classified by irreducible representations of the superconformal group $\mathrm{SU}(2,2|4)$. In particular, operators transforming in short representations, i.e. carrying a certain amount of supersymmetry, are protected from quantum corrections. Conversely, scaling dimensions of long superfields are subject to renormalization. In the supergravity approximation the correlator of two loops can be computed by evaluating the exchange amplitude of supergravity fields between two string worldsheets with loops as their boundaries [20]. Since supergravity fields are dual to protected operators transforming in short representations of the superconformal group, it is these operators that survive

[3] We thank T. Petkou for a discussion on this point.

in the local operator expansion (3.1) in the limit of large N and large 't Hooft coupling λ. However a closed Maldacena-Wilson loop completely breaks supersymmetry. Therefore its operator expansion at finite λ should certainly contain unprotected operators. We will show that this is indeed the case and that the operators with non-vanishing anomalous dimensions are the only ones coupling to the interacting (i.e. non-ladder) Feynman graphs in the limiting geometry we study.

We start with describing the lower dimensional operator content of the expansion (3.1) in field theory. There are only two types of gauge invariant operators of free field dimension two: the chiral primary operators (CPOs) O^a and the Konishi scalar K. Canonically normalized in the large N-limit they are given by

$$O^a = \frac{4\sqrt{2}\pi^2}{\lambda} C^a_{IJ} : \mathrm{Tr}(\Phi^I \Phi^J) : , \qquad K = \frac{4\pi^2}{\sqrt{3}\lambda} : \mathrm{Tr}(\Phi^I \Phi^I) : . \tag{3.5}$$

Here the traceless symmetric tensor C^a_{IJ} obeys $C^a_{IJ} C^b_{IJ} = \delta^{ab}$; $a = 1, \ldots, 20$. The protected operators O^a lie in a short supermultiplet and transform in the irrep **20** of the R-symmetry group SO(6). They have vanishing anomalous dimension. The Konishi scalar is the lowest component of the long supermultiplet [21] and it acquires an anomalous dimension in perturbation theory [22, 23]. In particular, its one-loop anomalous dimension is $\Delta^{(1)}_K = \frac{3\lambda}{4\pi^2}$ [23] which was extensively confirmed in recent studies [16]–[19].

Among the operators of classical dimension three we have to consider

$$: \mathrm{Tr}(\Phi^I \Phi^J \Phi^J) : , \qquad : \mathrm{Tr}(\Phi^{\{I} \Phi^J \Phi^{K\}}) : \tag{3.6}$$

and

$$J^I_{\mu\nu} =: \mathrm{Tr}(\Phi^I F_{\mu\nu}) : , \tag{3.7}$$

where $\{,\}$ stands for symmetrization with all traces removed. In particular the second operator in (3.6) is the protected CPO transforming in the **50** of SO(6). The leading contribution to the correlation function of W with any of the operators in (3.6) is of order $\lambda^3 \sim g^6$ and, therefore, in order to find the one-loop anomalous dimensions of these operators one has to analyze the order g^8 contribution of the correlator $\langle W\, W \rangle$, which is beyond our present knowledge. As to the operator $J^I_{\mu\nu}$, since the leading order behavior is $\langle W\, J^I_{\mu\nu} \rangle \sim \lambda^2$, we are able to use our perturbative results to deduce its anomalous scaling dimension.

The operator $J^I_{\mu\nu}$ is particularly interesting because its renormalization involves fermions. Indeed, the gauged 5-dimensional supergravity contains an antisymmetric tensor field $a_{\hat{\mu}\hat{\nu}}$ which is dual[4] to a dimension three tensor current $J^{I+}_{\mu\nu}$ belonging to the stress tensor multiplet of the boundary conformal field theory [24, 25]. This current is not a purely bosonic operator but also contains bilinear fermion terms. Hence, although in the free theory fermions do not couple to the Maldacena-Wilson loop, a naive operator $J^I_{\mu\nu}$ does not fit into the local operator expansion at strong coupling. Apparently we are encountering a splitting phenomenon which also occurs for operator product expansions of local operators [23, 17]: the free field operator $J^I_{\mu\nu}$ splits in perturbation theory into the

[4]In the formulation of $\mathcal{N} = 4$ SYM with Weyl fermions $a_{\hat{\mu}\hat{\nu}}$ couples to the YM operator $O^+_{\mu\nu}$ of reference [24, 25] transforming in the self-dual irrep $\mathbf{6}_c$ of SU(4).

sum of two operators, one of which is a protected operator whereas the other develops an anomalous scaling dimension and decouples at strong coupling.

To justify this picture define the following two operators

$$J^{I\,\pm}_{\mu\nu} = 2\,\mathrm{Tr}(\Phi^I F_{\mu\nu}) \pm \frac{1}{\sqrt{8}}\Sigma^I_{AB}\,\mathrm{Tr}(\bar{\psi}^A\gamma_{\mu\nu}\psi^B)\,. \tag{3.8}$$

Here Σ^I_{AB} is an antisymmetric matrix which intertwines the adjoint irrep of SO(4) with the fundamental of SO(6), $A, B = 1, \ldots, 4$. One can take for instance $\Sigma^I_{AB} = (\eta^k_{AB}, \bar{\eta}^k_{AB})$, where $k = 1, 2, 3$ and η^k_{AB} and $\bar{\eta}^k_{AB}$ are self-dual and anti-self dual 't Hooft symbols respectively. In free field theory the operators $J^{I\,\pm}_{\mu\nu}$ are orthogonal with respect to the two-point function and satisfy $4J^I_{\mu\nu} = J^{I\,+}_{\mu\nu} + J^{I\,-}_{\mu\nu}$. One also has[5]

$$\langle J^{I\,\pm}_{\mu\nu}(x) J^{J\,\pm}_{\lambda\rho}(y)\rangle = -\frac{\lambda^2\delta^{IJ}}{\pi^4(x-y)^6}\left[\delta_{\mu[\lambda}\delta_{\rho]\nu} + 2\frac{(x_\mu\delta_{\nu[\lambda} - x_\nu\delta_{\mu[\lambda})x_{\rho]}}{x^2}\right]. \tag{3.9}$$

In perturbation theory $J^{I\,+}_{\mu\nu}$ and $J^{I\,-}_{\mu\nu}$ have different renormalization group behavior, whereas $J^{I\,+}_{\mu\nu}$ belongs to the stress tensor multiplet and is non-renormalized, the operator $J^{I\,-}_{\mu\nu}$ has a non-vanishing anomalous scaling dimension.

The analysis of the dimension four operators is more involved because they also contain descendent fields, which are not orthogonal to primaries with respect to the two-point function. A given primary operator O contributes to the local operator expansion with all its derivative descendents. This contribution can be found by evaluating the correlation function of W with O

$$\frac{\langle W(C)\, O(0)\rangle}{\langle W\rangle} = \sum_k c_k R^{\Delta_k}\langle O^{(k)}(x)\, O(0)\rangle\,, \tag{3.10}$$

where k enumerates the infinite set of descendents of the primary operator O. In particular, the leading contributions of the descendents of the CPO, of the Konishi scalar and of the currents $J^{\pm}_{\mu\nu} = \theta^I J^{I\,\pm}_{\mu\nu}$ read as

$$\frac{\langle W(C)\, O^a(x)\rangle}{\langle W\rangle} = \kappa^a\frac{\lambda\sqrt{2}}{4}\frac{R^2}{(R^2+h^2)^2}\,, \tag{3.11}$$

$$\frac{\langle W(C)\, K(x)\rangle}{\langle W\rangle} = \frac{\lambda}{4\sqrt{3}}\frac{R^2}{(R^2+h^2)^2}\,, \tag{3.12}$$

$$\frac{\langle W(C)\, J^{\pm}_{\mu\nu}(x)\rangle}{\langle W\rangle} = \epsilon_{ij}\delta^i_\mu\delta^j_\nu\frac{i\lambda^2}{4\pi^2}\frac{R^3}{(R^2+h^2)^3}\,, \tag{3.13}$$

where $x = (0, 0, h, 0)$ and the loop of radius R is centered at the origin. Moreover we have defined $k_a = C^a_{IJ}\theta^I\theta^J$ so that $k_a k^a = 5/6$.

Expanding (3.11)–(3.13) in powers of R^2/h^2 one can identify every monomial as coming from a certain derivative descendent in the local operator expansion of $W(C)$. It is not difficult to find the form of these descendents in the free theory. For example to construct

[5]The fermionic propagator is $\langle\psi^{a,\,A}_\alpha(x)\bar{\psi}^{b,\,B}_\beta(y)\rangle = -i\delta^{AB}\frac{\delta^{ab}\gamma^\mu_{\alpha\beta}(x-y)_\mu}{2\pi^2(x-y)^4}$.

the dimension four descendents of the Konishi scalar one has to consider the following independent operators : $\partial_i\Phi^I\partial_i\Phi^I$:, : $\partial_{\bar{i}}\Phi^I\partial_{\bar{i}}\Phi^I$: and : $\partial_i^2\Phi^I\Phi^I$:. There is only one linear combination of these operators[6] which is orthogonal to K while the other two provide the descendents we are looking for. Since we have several derivative operators of the same free field dimension we expect that they will mix under renormalization. The correlation function of two Maldacena-Wilson loops to order g^6 does not contain enough information to establish their individual one-loop anomalous dimensions. We shall therefore restrict the further discussion to the operators of approximate dimension two and three mentioned above.

Summarizing the lower dimensional content of the local operator expansion we have

$$\frac{W(C)}{\langle W\rangle} = \mathbb{1} + R^{\Delta_K}\left[\frac{\lambda}{4\sqrt{3}}+\cdots\right]K(x) + R^{\Delta_O}\left[\frac{\lambda}{2\sqrt{2}}+\cdots\right]\kappa_a O^a(x) + \\ +R^{\Delta_+}\left[\frac{i\pi^2}{4}+\cdots\right]J^+_{ij}(x)\epsilon^{ij} + R^{\Delta_-}\left[\frac{i\pi^2}{4}+\cdots\right]J^-_{ij}(x)\epsilon^{ij} + \text{higher}\,. \quad (3.14)$$

Some comments are in order. In (3.14) the dots indicate higher order terms in λ of the corresponding operator expansion coefficients. Note that the coefficient of the CPO has been computed in [26] to all orders in λ under the assumption of vanishing radiative corrections. The scaling dimensions of the CPO and J^+_{ij} are $\Delta_O = 2$ and $\Delta_+ = 3$ respectively, while Δ_K and Δ_-, being the dimensions of the Konishi scalar and of the current J^-_{ij}, receive perturbative corrections. The operator content of (3.1) is sensitive to the orientation of the loop since the form ϵ^{ij} flips the sign under change of orientation. In the following it is convenient to distinguish different orientations by denoting a contour oriented clockwise by C^+ and the one with the opposite orientation by C^-.

By using eqs. (3.2), (3.14) and (3.11)–(3.13) we get in the limit $R_1 \to 0$ the following leading $\mathcal{O}(\lambda^2)$ contribution to the Green function of two Maldacena-Wilson loops:

$$\frac{\langle W(C_1)W(C_2^{\pm})\rangle_c}{\langle W\rangle^2} = \frac{\lambda^2}{48}R_1^{\Delta_K}\frac{R_2^2}{(R_2^2+h^2)^2} + \frac{5\lambda^2}{48}R_1^{\Delta_O}\frac{R_2^2}{(R_2^2+h^2)^2} \mp \\ \mp\frac{\lambda^2}{8}R_1^{\Delta_+}\frac{R_2^3}{(R_2^2+h^2)^3} \mp \frac{\lambda^2}{8}R_1^{\Delta_-}\frac{R_2^3}{(R_2^2+h^2)^3}\,. \quad (3.15)$$

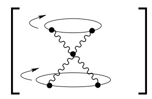
Here the first line represents the contribution from the Konishi scalar and the CPO while the second one is due to $J^{\pm}_{ij}$. Taking into account the one-loop anomalous dimension of the Konishi field we therefore obtain

$$\frac{\langle W(C_1)W(C_2^{\pm})\rangle_c}{\langle W\rangle^2} = \frac{\lambda^2}{8}\frac{R_1^2R_2^2}{(R_2^2+h^2)^2} \mp \frac{\lambda^2}{4}\frac{R_1^3R_2^3}{(R_2^2+h^2)^3} + \\ +\frac{\lambda^3}{64\pi^2}\frac{R_1^2R_2^2}{(R_2^2+h^2)^2}\log R_1 \mp \frac{\lambda^3}{8}\Delta_-^{(1)}\frac{R_1^3R_2^3}{(R_2^2+h^2)^3}\log R_1\,. \quad (3.16)$$

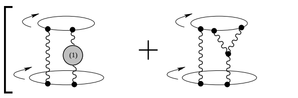
This formula predicts that the correlator (2.1) develops a logarithmic singularity in the limit $R_1 \to 0$ at order λ^3.

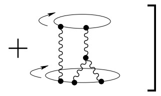
[6]In fact this is a primary operator coinciding with the component T_{ii} of the stress tensor $T_{\mu\nu}$ of free six bosons.

Now we are ready to compare the expected behavior of eq. (3.16), with the explicit two-loop calculation of [3]. We first note that in the limit considered, $R_1 \to 0$, only the interacting diagrams give rise to logarithmic terms, on which we shall focus. In agreement with the above predictions the contribution of the graphs can be worked out explicitly from the corresponding integral representations in this limit. We find for two loops of equal orientation the logarithmic contributions

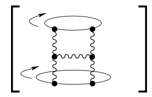


$$\Big._{R_1\to 0} = \frac{\lambda^3}{64\pi^2}\,\frac{R_1^3 R_2^3}{(R_2^2+h^2)^3}\log R_1\,,$$





$$\Big._{R_1\to 0} = \frac{\lambda^3}{32\pi^2}\,\frac{R_1^2 R_2^2}{(R_2^2+h^2)^2}\log R_1 - \frac{7\,\lambda^3}{64\pi^2}\,\frac{R_1^3 R_2^3}{(R_2^2+h^2)^3}\log R_1 - \frac{11\lambda^3}{96\pi^2}\,\frac{R_1^4 R_2^2}{(R_2^2+h^2)^3}\log R_1\,,$$



$$\Big._{R_1\to 0} = -\frac{\lambda^3}{64\pi^2}\,\frac{R_1^2 R_2^2}{(R_2^2+h^2)^2}\log R_1 + \frac{\lambda^3}{32\pi^2}\,\frac{R_1^3 R_2^3}{(R_2^2+h^2)^3}\log R_1 + \frac{59\lambda^3}{1152\pi^2}\,\frac{R_1^4 R_2^2}{(R_2^2+h^2)^3}\log R_1\,. \qquad (3.17)$$

The ladder graphs do not yield logarithmic contributions. Summing up we obtain

$$\frac{\langle W(C)W(C)\rangle_c}{\langle W\rangle^2} = \frac{\lambda^3}{64\pi^2}\frac{R_1^2R_2^2}{(R_2^2+h^2)^2}\log R_1 - \frac{\lambda^3}{16\pi^2}\frac{R_1^3R_2^3}{(R_2^2+h^2)^3}\log R_1 - \frac{73\lambda^3}{1152\pi^2}\frac{R_1^4R_2^2}{(R_2^2+h^2)^3}\log R_1\,. \qquad (3.18)$$

The result for graphs of opposite orientation differs from (3.17) only by a sign in front of the $(R_1R_2)^3\log R_1$ terms. The first term in the last expression indeed confirms the one-loop anomalous dimension of the Konishi field to be $\Delta_K^{(1)} = \frac{3\lambda}{4\pi^2}$. By comparing the second term in (3.18) with the local operator expansion predictions (3.16) we read off the one-loop anomalous dimension of the operator J_{ij}^{I-}

$$\Delta_-^{(1)} = \frac{\lambda}{2\pi^2}\,. \qquad (3.19)$$

Finally the last term in (3.18) carries information on the one-loop anomalous dimensions of the approximate dimension four operators, which are not orthogonal to the Konishi scalar and the CPO.

This completes our considerations of the perturbative local operator expansion. We conclude by emphasizing two observations concerning the above analysis. First, the interacting Feynman diagrams are the ones responsible for the appearance of unprotected operators in the local operator expansion. The ladder diagrams do not produce any logarithmic singularities (at the level of the perturbation theory we are working at) and contribute only to the local operator expansion coefficients. In this respect it would be interesting to understand whether the interacting graphs also contribute to the local operator expansion coefficient of the CPOs. If this is not the case this would mean that interacting graphs couple solely to the unprotected operators of the theory, like the Konishi field. At strong coupling the Konishi field decouples from the theory, which would imply in turn that the sector of interacting graphs coupling to it becomes negligible in comparison to the ladders.[7] Unfortunately the information we have at our disposal so far does not suffice to answer this question. Second, in order to match the supergravity picture the operator expansion of the bosonic Maldacena-Wilson loop should contain renormalized operators which are constructed from bosons and fermions. Fermions emerge due to the splitting mechanism and at the level of perturbation theory this can be viewed as a sign of supersymmetry in the theory of a purely bosonic operator.

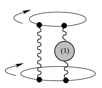
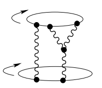
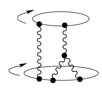
4. New results on the matrix model conjecture

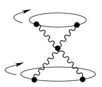
Armed with the complete order g^6 result for the connected correlator of two Maldacena-Wilson loops of equal orientation, we may now perform a further test of the Drukker-Gross matrix model conjecture [4]. Based on an anomaly argument these authors argue that the expectation value of an arbitrary number of coincident, possibly multiply wound, circular Maldacena-Wilson loops of equal orientation is given by solving a purely combinatorial problem.[8] The latter is easily seen to correspond to computing correlators in a simple gaussian matrix model. If correct this conjecture amounts to the claim that all interacting graphs contributing to the loop correlator vanish.

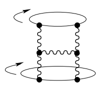
The relevant limit in order to extract the result of two incident circles of equal orientation from our geometry is

$$R_1 = R_2 = R \qquad \text{and} \qquad h \to 0\,. \tag{4.1}$$

Given the explicit integral representations for individual diagrams found in [3] it is straightforward to analytically perform this limit. Remarkably, and quite non-trivially, one finds after some calculation that *all* interacting non-ladder graphs cancel:

$0 = \Big[$  $+$  $+$  $\Big]_{\substack{R_1=R_2 \\ h\to 0}}$

$0 = \Big[$  $\Big]_{\substack{R_1=R_2 \\ h\to 0}}$

$0 = \Big[$  $\Big]_{\substack{R_1=R_2 \\ h\to 0}}$ (4.2)

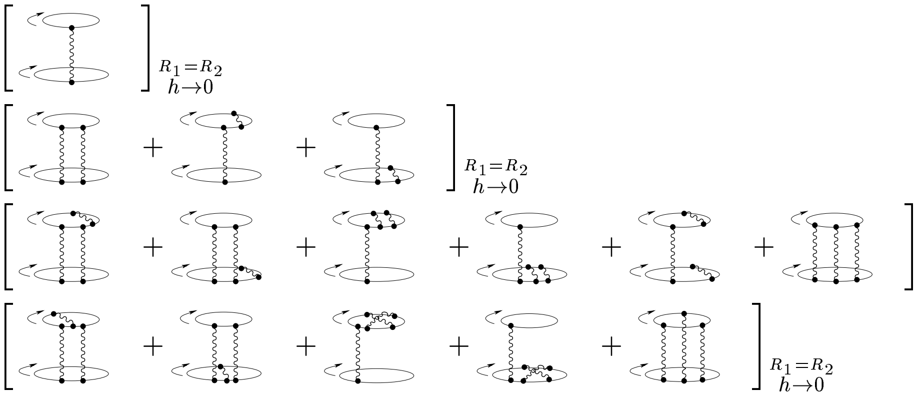
Therefore only the contribution of the ladder-diagrams survives. This result represents a two loop order test of the Drukker-Gross conjecture [4]. For the ladder-graphs one has in

[7] On the other hand, the sector of interacting graphs coupling to unprotected operators of *finite* anomalous dimension at strong coupling will survive.

[8] This more general problem is equivalent to finding the expectation value of a single circular Wilson loop in an arbitrary representation of the gauge group.

the limit (4.1)

$$\frac{\lambda}{4} = \left[\cdots \right]_{\substack{R_1=R_2\\ h\to 0}}, \qquad \frac{3\,\lambda^2}{32} = \left[\cdots \right]_{\substack{R_1=R_2\\ h\to 0}}, \qquad \frac{5\,\lambda^3}{384} = \left[\cdots \right]_{\substack{R_1=R_2\\ h\to 0}}, \qquad \frac{\lambda^3}{192\,N^2} = \left[\cdots \right]_{\substack{R_1=R_2\\ h\to 0}} \tag{4.3}$$

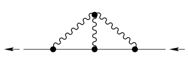
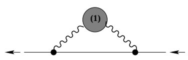


which indeed follows from the proposed matrix model description [12, 27]

$$\begin{aligned}\langle \mathcal{W}(C_1)\, \mathcal{W}(C_2)\rangle_c &= \frac{1}{Z}\int dM\ \mathrm{Tr}\, e^M\ \mathrm{Tr}\, e^M\, e^{-\frac{2N}{\lambda}\ \mathrm{Tr}\, M^2} - \Big(\frac{1}{Z}\int dM\ \mathrm{Tr}\, e^M\, e^{-\frac{2N}{\lambda}\ \mathrm{Tr}\, M^2}\Big)^2 \\ &= \frac{\sqrt{\lambda}}{2} I_0(\sqrt{\lambda}) I_1(\sqrt{\lambda}) + \frac{\lambda^2}{192\,N^2}\Big(3I_1(\sqrt{\lambda})^2 + I_2(\sqrt{\lambda})[2I_0(\sqrt{\lambda}) + I_2(\sqrt{\lambda})]\Big) + \\ &\quad + \mathcal{O}\left(\frac{1}{N^4}\right),\end{aligned} \tag{4.4}$$

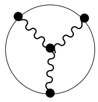
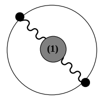
where $I_n(x)$ is the nth order modified Bessel function. Expanding (4.4) reproduces our findings to $\mathcal{O}(\lambda^3)$.

Let us review the existing perturbative evidence for the matrix model conjecture. Clearly, performing the sum of all ladder graphs for correlators of incident Maldacena-Wilson circles may be rephrased by a gaussian matrix model of the form (4.4). This is due to the fact that the combined gluon-scalar propagator trivializes to a constant if both of its ends lie on a circle. An alternative way of understanding this starts from the observation that the ladder graphs of an infinite Maldacena-Wilson line are identically zero. Under an inversion the line is mapped to a circle, so one might expect the correlators of the line and the circle to be identical by conformal invariance of the theory. However, as we know, this is not the case, and as shown in [4] the discrepancy is due to additional total derivative terms that the gluon propagator picks up under an inversion. These total derivatives let the contribution from the *ladder* diagrams collapse to a point and turn the circle correlator into a correlator in a zero dimensional field theory, the matrix model (4.4). The story for the interacting graphs is, however, considerably more intricate.[9] Now at least one leg of the modified propagator is attached to an interaction vertex which may no longer be simply integrated by parts. An explicit example where such a graph by graph consideration fails appears already at the one loop ($\mathcal{O}(g^4)$) level [2]. Here the interacting graphs for the infinite Maldacena-Wilson line vanish separately

 $= 0$  $= 0$ (4.5)

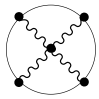
[9]Here we cannot follow the arguments of [4] which claim that also the interacting sector collapses to a point. Even stronger, the authors speculate that there are no contributions from the interacting graphs to all orders in perturbation theory.

due to the appearance of the standard combination $\dot{x}_i \cdot \dot{x}_j - |\dot{x}_i|\,|\dot{x}_j|$ in the numerators, which vanishes for straight line trajectories. Their image graphs under inversion, however, do not vanish and are actually individually divergent in four dimensions [2]

 $= \text{divergent}$  $= \text{divergent}\,.$ (4.6)

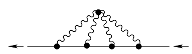
It is only that the sum of these two graphs vanishes, due to the subtle cancellation of bulk (right diagram) versus boundary (left diagram) divergences for the Maldacena-Wilson loop operator. Given this, one might speculate on the existence of a modified anomaly argument, in which one only considers the inversion of classes of individually divergent graphs, which sum to a finite result. This is indeed a necessity in view of the fact that the dimensional regularization needed for divergent graphs breaks conformal invariance. If true and if the resulting matrix model is indeed gaussian, such a modified conjecture would claim that the image graph under inversion of a vanishing interacting graph on the line should either be zero or divergent. Indeed the results of [2] in (4.6) and our reported two-loop findings of (4.2) support this statement.

However, let us now give a counterexample to such a modified anomaly conjecture. Consider the four point insertion on the line and the circle, an order g^6 interacting graph. On the circle it takes the form

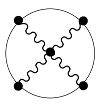


$$= \frac{g^6 N^2 (N-1)}{8} \int \frac{d^4 z}{(2\pi)^8} \int_0^{2\pi} d\tau_1 \int_0^{\tau_1} d\tau_2 \int_o^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 \, \times$$
$$\times \frac{2\cdot(2,4)\cdot(3,1) - (1,4)\cdot(2,3) - (1,2)\cdot(3,4)}{(x_1-z)^2\,(x_2-z)^2\,(x_3-z)^2\,(x_4-z)^2}\,, \qquad (4.7)$$

where we have parametrized the four points on the circle by $x_i = (\cos\tau_i, \sin\tau_i, 0, 0)$ and introduced the notation $(i,j) := \dot{x}_i \cdot \dot{x}_j - |\dot{x}_i|\,|\dot{x}_j|$. From this structure one immediately deduces the vanishing of the corresponding graph on the line

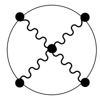
 $= 0$ (4.8)

due to $(i,j)|_{\text{line}} = 0$. The circle graph (4.7) on the other hand may be brought into the form



$$= \frac{g^6 N^2 (N-1)}{8\,(2\pi)^6} \int_0^{\infty} dz \int_0^{\infty} d\rho \int_0^{2\pi} d\tau_1 \int_0^{\tau_1} d\tau_2 \int_o^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 \, \times$$
$$\times \rho\, z\, \frac{2\cdot[2,4]\cdot[3,1] - [1,4]\cdot[2,3] - [1,2]\cdot[3,4]}{\prod_{i=1}^{4}(1+\rho^2+z^2-2z\cos\tau_i)}\,, \qquad (4.9)$$

where $[i,j] := (i,j)|_{\text{circle}} = 1 - \cos(\tau_i - \tau_j)$. We claim that this integral is finite. Unfortunately we have not yet been able to determine its value analytically, but could evaluate it to a high degree of accuracy numerically. We find that



$$= \frac{g^6 N^2 (N-1)}{2^7} \cdot 9.25(47) \cdot 10^{-3} \simeq \frac{g^6 N^2 (N-1)}{(4!)^3}\,, \qquad (4.10)$$

where the last value represents our sophisticated guess for the exact value of the integral.

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This consideration shows that the structure of the interacting graphs at order g^6 for a single Maldacena-Wilson loop has a much richer structure than one would expect from the g^4 results. Although we have only considered a single graph, its finiteness casts some doubt on the conjecture [4] that the circular loop can be obtained by simply solving a combinatorial problem, described by a gaussian matrix model. Since it is far from clear how to obtain the above X-graph by pure combinatorial reasoning it is, furthermore, not obvious whether introducing interactions into the matrix model might help. Of course an individual graph such as the one we just studied is not gauge invariant by itself, but that is also true for the artificial — in the context of non-abelian gauge theory — restriction to ladder graphs. Nevertheless, it might still be true that the *sum* of all vertex graphs, in Feynman gauge, cancels. If so, we feel we have demonstrated that the true mechanism is wanting. Clearly it would be highly desirable to know the complete $\mathcal{O}(g^6)$ result and to really understand the behavior of the interacting graphs under the inversion.

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